Barrier Options
Pricing A Down-And-Out Call Option

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We want to build a model to value the following barrier option...

Down-And-Out Call Option: Call option that can be exercised at time T provided that it doesn't knock-out prior to that time. The knock-out occurs if stock price crosses some barrier value (barrier value is less than stock price at time zero) sometime over the time interval $[0, T]$.

To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to price barrier call options given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
S	Stock price at time zero	\$20.00
K	Exercise price	\$18.00
B	Barrier price	\$15.00
T	Option expiration in years	2.00
μ	Annual risk-free rate (%)	5.00
ϕ	Annual dividend yield (%)	0.00
σ	Annual return volatility (%)	30.00

Our task is to answer the following question...

Question: What is the time zero value of the down-and-out call option?

Problem Setup

We will define the variables α and υ to be expected return mean and expected return variance, respectively, over the time interval $[0, T]$. Using the model parameters in Table 1 above we will define the following variables...

$$\alpha = \left(\mu - \phi - \frac{1}{2}\sigma^2\right)T \text{ ...and... } \upsilon = \sigma^2 T \tag{1}$$

We will define the variable m to be the minimum value of the random return over the time interval $[0, T]$ and the variable w to be the random return at time T. We will define the function $a(m, w)$ to be the joint distribution function of m and w. Using Equation (1) above the equation for the joint distribution function is... [1]

$$a(m, w) = \frac{2\,(w - 2m)}{\upsilon\sqrt{2\pi\upsilon}} \operatorname{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\upsilon}(2m - w)^2\right\} \tag{2}$$

Since we will be working directly with the Brownian motion we need to know the point at which the option is at-the-money (i.e. stock price equals option exercise price) and the point at which the barrier is crossed (i.e. stock price equals option barrier price). Using the model parameters in Table 1 above we will define the following variables...

$$\text{Threshold value: } x = \ln\left(\frac{K}{S}\right) \text{ ...and... Barrier value: } y = \ln\left(\frac{B}{S}\right) \tag{3}$$

We will define the variable C_t to be call option value at time $t \leq T$. The value of the call option is the expected call payoff at time T under the risk-neutral probability measure discounted at the risk-free rate. Using the model parameters in Table 1 above the equation for call option value at time zero is...

$$C_0 = \mathbb{E}^Q \left[\text{Exp}\left\{ -\mu T \right\} \text{Max}\left\{ S_T - K, 0 \right\} \right] \text{ ...where... } Q = \text{risk-neutral measure} \tag{4}$$

Down-And-Out Call Option Value

Using Equations (2), (3) and (4) above and the model parameters in Table 1 above the equation for the value of a down-and-out call option at time zero is...

$$\begin{aligned} C_0 = & \int_{w=x}^{w=0} \int_{m=y}^{m=w} \left(S\,\text{Exp}\left\{ w \right\} - K \right) \text{Exp}\left\{ -\mu T \right\} a(m,w)\, \delta m\, \delta w \\ & + \int_{w=0}^{w=\infty} \int_{m=y}^{m=0} \left(S\,\text{Exp}\left\{ w \right\} - K \right) \text{Exp}\left\{ -\mu T \right\} a(m,w)\, \delta m\, \delta w \end{aligned} \tag{5}$$

We can rewrite Equation (5) above as...

$$C_0 = S\,\text{Exp}\left\{ -\mu T \right\} I_1 - K\,\text{Exp}\left\{ -\mu T \right\} I_2 + S\,\text{Exp}\left\{ -\mu T \right\} I_3 - K\,\text{Exp}\left\{ -\mu T \right\} I_4 \tag{6}$$

where...

$$I_1 = \int_{w=x}^{w=0} \int_{m=y}^{m=w} \text{Exp}\left\{ w \right\} a(m,w)\, \delta m\, \delta w \tag{7}$$

$$I_2 = \int_{w=x}^{w=0} \int_{m=y}^{m=w} a(m,w)\, \delta m\, \delta w \tag{8}$$

$$I_3 = \int_{w=0}^{w=\infty} \int_{m=y}^{m=0} \text{Exp}\left\{ w \right\} a(m,w)\, \delta m\, \delta w \tag{9}$$

$$I_4 = \int_{w=0}^{w=\infty} \int_{m=y}^{m=0} a(m,w)\, \delta m\, \delta w \tag{10}$$

How do we know that Equation (5) above represents the true value of the call? In that equation we use the variable w to represent the value of the Brownian motion at time T and the variable m to represent the minimum value of the Brownian motion over the time interval [0, T]. Lets take a look at the first double integral...

$$\int_{w=x}^{w=0} \int_{m=y}^{m=w} \left(S\,\text{Exp}\left\{ w \right\} - K \right) \text{Exp}\left\{ -\mu T \right\} a(m,w)\, \delta m\, \delta w \tag{11}$$

The bounds of integration for the first integral is $w = x$ to $w = 0$. Here we want to weight possible stock prices at time T as stock price goes from the exercise price (point x in the Brownian motion) to stock price at time zero (point 0 in the Brownian motion). Given that we want to know the probabilities that stock price will not cross the minimum barrier and the fact that the minimum of the Brownian motion must be less than stock price the bounds of integration for the second integral are $m = y$, which represents the barrier, to $w = 0$, which represents current stock price. The probability of the w and m combination is $a(m,w)\, \delta m\, \delta w$.

Lets now take a look at the second double integral...

$$\int_{w=0}^{w=\infty} \int_{m=y}^{m=0} \left(S\,\text{Exp}\left\{ w \right\} - K \right) \text{Exp}\left\{ -\mu T \right\} a(m,w)\, \delta m\, \delta w \tag{12}$$

For this integral we want to know the w and m combinations where stock price at time t is greater than stock price at time zero.

Solution To The First Integral [2]

Using Appendix Equation (31) below we can rewrite the first integral (Equation (7) above) as...

$$I_1 = \int_{w=x}^{w=0} \int_{m=y}^{m=w} \mathrm{Exp}\left\{w\right\} a(m,w)\,\delta w = \int\limits_{w=x}^{w=0} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=y}^{m=w} \delta w \tag{13}$$

Using Appendix Equation (39) below we can rewrite the first half of the integral in Equation (13) above as...

$$I_{1a} = \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2}\right\}\Bigg[CNDF(0,\alpha+\upsilon,\upsilon) - CNDF(x,\alpha+\upsilon,\upsilon)\Bigg] \tag{14}$$

Using Appendix Equation (37) below we can rewrite the second half of the integral in Equation (13) above as...

$$I_{1b} = \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2} + 2\,y + \frac{2\,y\,\alpha}{\upsilon}\right\}\Bigg[CNDF(0,\alpha+\upsilon+2\,y,\upsilon) - CNDF(x,\alpha+\upsilon+2\,y,\upsilon)\Bigg] \tag{15}$$

Using Equations (14) and (15) above and the model parameters in Table 1 above the solution to Equation (13) above is...

$$I_1 = I_{1a} - I_{1b} = 0.092807 - 0.060803 = 0.032005 \tag{16}$$

Solution To The Second Integral [2]

Using Appendix Equation (30) below we can rewrite the second integral (Equation (8) above) as...

$$I_2 = \int_{w=x}^{w=0} \int_{m=y}^{m=w} a(m,w)\,\delta m\,\delta w = \int\limits_{w=x}^{w=0} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=y}^{m=w} \delta w \tag{17}$$

Using Appendix Equation (35) below we can rewrite the first half of the integral in Equation (17) above as...

$$I_{2a} = CNDF(0,\alpha,\upsilon) - CNDF(x,\alpha,\upsilon) \tag{18}$$

Using Appendix Equation (33) below we can rewrite the second half of the integral in Equation (17) above as...

$$I_{2b} = \mathrm{Exp}\left\{\frac{2\,y\,\alpha}{\upsilon}\right\}\Bigg[CNDF(0,\alpha+2\,y,\upsilon) - CNDF(x,\alpha+2\,y,\upsilon)\Bigg] \tag{19}$$

Using Equations (18) and (19) above and the model parameters in Table 1 above the solution to Equation (17) above is...

$$I_2 = I_{2a} - I_{2b} = 0.097751 - 0.064188 = 0.033563 \tag{20}$$

Solution To The Third Integral [2]

Using Appendix Equation (31) below we can rewrite the third integral (Equation (9) above) as...

$$I_3 = \int_{w=0}^{w=\infty} \int_{m=y}^{m=0} \mathrm{Exp}\left\{w\right\} a(m,w)\,\delta w = \int\limits_{w=0}^{w=\infty} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=y}^{m=0} \delta w \tag{21}$$

Using Appendix Equation (37) below we can rewrite the first half of the integral in Equation (21) above as...

$$\begin{aligned} I_{3a} &= \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2} + 2\,(0) + \frac{2\,(0)\,\alpha}{\upsilon}\right\}\Bigg[CNDF(\infty,\alpha+\upsilon+2\,(0),\upsilon) - CNDF(0,\alpha+\upsilon+2\,(0),\upsilon)\Bigg] \\ &= \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2}\right\}\Bigg[1 - CNDF(0,\alpha+\upsilon+2\,(0),\upsilon)\Bigg] \end{aligned} \tag{22}$$

Using Appendix Equation (37) below we can rewrite the second half of the integral in Equation (21) above as...

$$I_{3b} = \text{Exp}\left\{\alpha + \frac{v}{2} + 2\,y + \frac{2\,y\,\alpha}{v}\right\}\left[CNDF(\infty, \alpha + v + 2\,y, v) - CNDF(0, \alpha + v + 2\,y, v)\right]$$
$$= \text{Exp}\left\{\alpha + \frac{v}{2} + 2\,y + \frac{2\,y\,\alpha}{v}\right\}\left[1 - CNDF(0, \alpha + v + 2\,y, v)\right] \tag{23}$$

Using Equations (22) and (23) above and the model parameters in Table 1 above the solution to Equation (21) above is...

$$I_3 = I_{3a} - I_{3b} = 0.743629 - 0.188296 = 0.555334 \tag{24}$$

Solution To The Fourth Integral [2]

Using Appendix Equation (30) below we can rewrite the fourth integral (Equation (10) above) as...

$$I_4 = \int_{w=0}^{w=\infty}\int_{m=y}^{m=0} a(m,w)\,\delta m\,\delta w = \int\limits_{w=0}^{w=\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{\frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2\,v}(2\,m - w)^2\right\}\Bigg[_{m=y}^{m=0}\delta w \tag{25}$$

Using Appendix Equation (33) below we can rewrite the first half of the integral in Equation (25) above as...

$$I_{4a} = \text{Exp}\left\{\frac{2\,(0)\,\alpha}{v}\right\}\left[CNDF(\infty, \alpha + 2\,(0), v) - CNDF(0, \alpha + 2\,(0), v)\right]$$
$$= 1 - CNDF(0, \alpha, v) \tag{26}$$

Using Appendix Equation (33) below we can rewrite the second half of the integral in Equation (25) above as...

$$I_{4b} = \text{Exp}\left\{\frac{2\,y\,\alpha}{v}\right\}\left[CNDF(\infty, \alpha + 2\,y, v) - CNDF(0, \alpha + 2\,y, v)\right]$$
$$= \text{Exp}\left\{\frac{2\,y\,\alpha}{v}\right\}\left[1 - CNDF(0, \alpha + 2\,y, v)\right] \tag{27}$$

Using Equations (26) and (27) above and the model parameters in Table 1 above the solution to Equation (25) above is...

$$I_{4d} = I_{4a} - I_{4b} = 0.509402 - 0.148176 = 0.361227 \tag{28}$$

The Answer To The Pricing Question

The values of x, y, α and v per Equations (1) and (3) above are...

$$\begin{array}{lllllll}
x & = & \ln(K \div S) & = & \ln(18.00 \div 20.00) & = & \text{-0.1054} \\
y & = & \ln(B \div S) & = & \ln(16.00 \div 20.00) & = & \text{-0.2231} \\
\alpha & = & (\mu - \phi - 0.50 \times \sigma^2)T & = & (0.05 - 0.00 - 0.50 \times 0.30^2) \times 2.00 & = & 0.0100 \\
v & = & \sigma^2 \times T & = & 0.30^2 \times 2.00 & = & 0.1800
\end{array}$$

The values of the integrals per Equations (16), (20), (24) and (28) above are...

$$\begin{array}{lllll}
I_1 & = & f(x,y,\alpha,v) & = & 0.032005 \\
I_2 & = & f(x,y,\alpha,v) & = & 0.033563 \\
I_3 & = & f(x,y,\alpha,v) & = & 0.555334 \\
I_4 & = & f(x,y,\alpha,v) & = & 0.361227
\end{array}$$

The present value of current stock price and exercise price are...

PV Stock price	=	$S\,e^{-rt}$	=	$20.00 \times exp(-0.05 \times 2.00)$	=	18.10
PV Exercise price	=	$K\,e^{-rt}$	=	$18.00 \times exp(-0.05 \times 2.00)$	=	16.29

The value of the down-and-out call option per Equation (6) above is...

$$C_0 = Se^{-rt}I_1 - Ke^{-rt}I_2 + Se^{-rt}I_3 - Ke^{-rt}I_4$$
$$= (18.10)(0.032005) - (16.29)(0.033563) + (18.10)(0.555334) - (16.29)(0.361227)$$
$$= 4.20 \tag{29}$$

References

[1] Integral Solutions - Integral Two - Anti-Derivative of the Joint Density Function, Schurman

[2] Integral Solutions - Integral Four - Barrier Integral Solutions, Schurman

Appendix

Base Equations

The equation for the integral of the joint density function as defined by Equation (2) above is...

$$I_1 = \int\limits_{w=c}^{w=d} \int\limits_{m=a}^{m=b} a(m,w)\,\delta m\,\delta w = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=a}^{m=b}\,\delta w \tag{30}$$

The equation for the integral of the product of the exponential of the random variable w and the joint density function as defined by Equation (2) above is...

$$I_2 = \int\limits_{w=c}^{w=d} \int\limits_{m=a}^{m=b} \mathrm{Exp}\left\{w\right\} a(m,w)\,\delta w = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=a}^{m=b}\,\delta w \tag{31}$$

A. Using Base Equation (30) above we want to solve the following integral...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\delta w \ \text{ ...where... } \ m = \text{constant} \tag{32}$$

The solution to the integral in Appendix Equation (32) above is...

$$I = \mathrm{Exp}\left\{\frac{2\,m\,\alpha}{\upsilon}\right\}\Bigg[CNDF(d,\alpha+2\,m,\upsilon) - CNDF(c,\alpha+2\,m,\upsilon)\Bigg] \tag{33}$$

B. Using Base Equation (30) above we

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\delta w \ \text{ ...where... } \ m = \text{random variable } w \tag{34}$$

The solution to the integral in Appendix Equation (34) above is...

$$I = CNDF(d,\alpha,\upsilon) - CNDF(c,\alpha,\upsilon) \tag{35}$$

C. Using Base Equation (31) above we

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\delta w \ \text{ ...where... } \ m = \text{constant} \tag{36}$$

The solution to the integral in Appendix Equation (36) above is...

$$I = \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2} + 2\,m + \frac{2\,m\,\alpha}{\upsilon}\right\}\Bigg[CNDF(d,\alpha+\upsilon+2\,m,\upsilon) - CNDF(c,\alpha+\upsilon+2\,m,\upsilon)\Bigg] \tag{37}$$

D. Using Base Equation (31) above we

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\delta w \ \text{ ...where... } \ m = \text{random variable } w \tag{38}$$

The solution to the integral in Appendix Equation (38) above is...

$$I = \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2}\right\}\Bigg[CNDF(d,\alpha+\upsilon,\upsilon) - CNDF(c,\alpha+\upsilon,\upsilon)\Bigg] \tag{39}$$